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Normed Linear Spaces and Optimal Design of Discrete Systems

P. WEIDNER

Zentralinstitut für Angewandte Mathematik der Kernforschungsanlage Jülich, Jülich, Germany

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SUMMARY
The behaviour of discrete mechanical or electrical systems under the action of disturbances shall be weighted. In a space which contains the solutions of the corresponding differential equations appropriate norms are introduced. The optimal design problem can then be interpreted as an approximation problem for the zero solution. A method for the delivering from the initial conditions is proposed.

1. Introduction

Designing a mechanical or electrical system the first requirement is that of stability. This means that the perturbed system must go back to his equilibrium state or position. Beyond this it is desirable that the system behaves "as well as possible" under the actions of perturbations, it should reach its equilibrium state in shortest possible time, or the occurring amplitudes should be as small as possible and so on. In contrast to control problems where one compensates disturbances of a given system from without, the system in our problem stabilizes itself. It is the aim of this paper to give a systematic treatment and some mathematical aspects of the optimization problem in the above sense.

2. Mathematical Preliminaries

We recall the concept of a linear space [2]. This is a set with an associative and commutative addition and a multiplication with real numbers together with the usual distributive laws.

In such spaces we consider norms, indicated as $\| \;\|$, this are functionals which obey the following rules:

$$\begin{aligned} &\text{i)}\ \|a\| \geqq 0\,, \quad \|a\| = 0 \text{ iff } a=0 \\ &\text{ii)}\ \|\alpha a\| = |\alpha| \cdot \|a\| \\ &\text{iii)}\ \|a+b\| \leqq \|a\| + \|b\|\,. \end{aligned} \tag{1}$$

Here a and b are elements of the linear space, α is a real number.

A linear space together with a norm is called a normed linear space.

As examples we call to mind the space R^n of n-vectors $\boldsymbol{a} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where the following norms can be introduced:

$$\|\boldsymbol{a}\|_p = (|\alpha_1|^p + \ldots + |\alpha_n|^p)^{1/p}\,, \qquad p=1, 2, \ldots\,. \tag{2}$$

$$\|\boldsymbol{a}\|_\infty = \max_{1 \leqq i \leqq n} |\alpha_i|\,. \tag{3}$$

A further example is the space $C[\alpha, \beta]$ of functions f continuous in the closed interval $[\alpha, \beta]$ together with one of the norms

$$\|f\|_p = \left(\int_\alpha^\beta |f(t)|^p dt \right)^{1/p}, \qquad p=1, 2, \ldots \tag{2a}$$

$$\|f\|_\infty = \max_{\alpha \leqq t \leqq \beta} |f(t)|\,. \tag{3a}$$

Given r normed linear spaces L_i with norms $\| \|^{L_i}$ $(i=1, 2, \dots, r)$ it is possible to construct a new linear space $L=L_1 \times L_2 \times \dots \times L_r$, the product space of $L_1, L_2, \dots, L_r$. The elements of L are r-tuples of elements from the spaces L_i respectively. For example the space R^n can be interpreted as the n-fold product of the linear space R, the space of real numbers. In analogy to R^n one can introduce in L the norms

$$\|\boldsymbol{a}\|_p^L = [(\|a_1\|^{L_1})^p + \dots + (\|a_r\|^{L_r})^p]^{1/p}, \qquad p=1, 2, \dots \tag{2b}$$

$$\|\boldsymbol{a}\|_\infty^L = \max_{1 \leq i \leq r} \|a_i\|^{L_i}. \tag{3b}$$

Here a_i is a element of L_i and $\boldsymbol{a}=(a_1, a_2, \dots a_r)$ denotes a element of the product space L.

In a normed linear space infinitesimal operations can be carried out and the notions of convergence, completeness, continuity and so on are understood in the usual way, always with respect to the particular norm.

3. Normed Linear Spaces and Discrete Systems

Discrete mechanical or electrical systems are generally described through systems of ordinary differential equations. Every such system can be transformed to a first order system [1]

$$\dot{x}_i = X_i(x_1, x_2, \dots, x_n, t), \qquad i=1, 2, \dots, n. \tag{4}$$

Besides there are initial conditions

$$x_i(t=0) = x_{i0}. \tag{5}$$

We look now upon a space of which the solutions of these equations are elements. It is known from the theory of differential equations [1] that the solutions are continuous functions, therefore they are elements of the space

$$C^n = C^n[0, \infty) = \underbrace{C[0, \infty) \times C[0, \infty) \times \dots \times C[0, \infty)}_{n\text{-times}}$$

of vector-valued functions, continuous in the interval $[0, \infty)$. With our general methods for the construction of norms in product spaces we can introduce in C^n many norms of whom we give in the following some examples.

(a) As norm in $C[0, \infty)$ we take (3a)

$$\|x_i\|_\infty = \sup_{0 \leq t < \infty} |x_i(t)| \qquad i=1, 2, \dots, n.$$

We must write sup instead of max because the interval is not closed. Further we must suppose that the expression on the right-hand side exists, this means $x_i(t)$ must be bounded. With (3b) we then construct the norm

$$\|\boldsymbol{x}\|_{\infty,\infty} = \max_{1 \leq i \leq n} \sup_{0 \leq t < \infty} |x_i(t)| \tag{6}$$

in C^n.

$\boldsymbol{x}$ is the solution vector $\boldsymbol{x}(t) = (x_1(t), \dots, x_n(t))$.

The first index on the norm refers to the norm in $C[0, \infty)$, the second to the construction method in C^n corresponding to (2b) or (3b).

(b) Now we choose as norm in $C[0, \infty)$ (2a) with $p=2$

$$\|x_i\|_2 = \left(\int_0^\infty x_i^2(t)\,dt\right)^{\frac{1}{2}}.$$

We must make here a further restriction upon $x_i(t)$, the integral must exist. With (3b) and $p=2$ we then have as norm in C^n

$$\|\boldsymbol{x}\|_{2,2}=\left(\sum_{i=1}^{n}\int_{0}^{\infty}x_i^2(t)dt\right)^{\frac{1}{2}}. \tag{7}$$

This expression is analogous to the quadratic control surface in control theory.

(c) We get an expression corresponding to the linear control surface if we take as $C[0, \infty)$-norm (2a) with $p=1$

$$\|x_i\|_1=\int_0^{\infty}|x_i(t)|dt$$

and for C^n (2b) with $p=1$

$$\|\boldsymbol{x}\|_{1,1}=\sum_{i=1}^{n}\int_0^{\infty}|x_i(t)|dt. \tag{8}$$

(d) Beside these and similar norms it is still possible to construct many other norms with the aid of different weight functions which must be continuous and positive in $[0, \infty)$. The time weighted control functions are the most important examples.

4. Optimal Design of Systems

The right-hand sides of equations (4) depend not only of $x_1, x_2, \ldots, x_n$ and t, but also of some parameters of the system, such as spring and damping parameters or impedances. These quantities can themselves be functions of the variables $x_1, \ldots, x_n$ or the time t. The behaviour of a system will depend of the size of these parameters.

We suppose the system (4) to be in such a form that the function $\boldsymbol{x}\equiv\boldsymbol{0}$ corresponds to the equilibrium state with regard to which the behaviour of the system shall be examined.

When disturbances characterized through the initial conditions (5) act on the system, the deviation from the state $\boldsymbol{x}\equiv\boldsymbol{0}$ should be small in some sense. If the above mentioned parameters are chosen in such a way that for fixed initial values the state $\boldsymbol{x}\equiv\boldsymbol{0}$ is best approximated in a certain norm by the solution of (4), we call the system optimal with respect to this norm. For optimal design we have thus to vary the parameters till the expression $\|\boldsymbol{x}-\boldsymbol{0}\|=\|\boldsymbol{x}\|$ is a minimum. This is a nonlinear approximation problem for the function identical to zero.

Here we must point out two facts. First, the optimal solution will always depend on the chosen norm, and secondly, it will depend on the initial values. To illustrate the second fact we treat the very simple example of a damped vibrating point mass. We set the mass and the spring constant to unity, the damping constant is $2D$, x is the deflection of the mass point. The equation in the usual form reads

$$\ddot{x}+2D\dot{x}+x=0$$

with initial conditions

$$x(0)=x_0, \quad \dot{x}(0)=\dot{x}_0.$$

In our terminology of equations (4) we have with

$$\begin{aligned}\boldsymbol{x}&=(x_1, x_2)=(x, \dot{x})\\ \dot{x}_1&=x_2\\ \dot{x}_2&=-2Dx_2-x_1\end{aligned} \tag{9}$$

and

$$x_1(0)=x_{10}, \quad x_2(0)=x_{20}.$$

For the optimization we take the norm (7)

$$(\|\boldsymbol{x}\|_{2,2})^2=\int_0^{\infty}[x_1^2(t)+x_2^2(t)]dt$$

and obtain

$$(\|x\|_{2,2})^2 = x_{10}^2 D + x_{10}x_{20} + \frac{1}{2D}(x_{10}^2 + x_{20}^2). \tag{10}$$

D is the only parameter to be varied. For the minimum value of (10) this gives

$$D^2 = \frac{x_{10}^2 + x_{20}^2}{2x_{10}^2}. \tag{11}$$

This shows the dependence of the initial conditions. But there is yet another difficulty. For $x_{20}=0$ we obtain from (11) the reasonable value $D=1/\sqrt{2}$. But for x_{10} tending to zero, D tends to infinity. The solution for this limiting case is no longer continuous. This result is a consequence of the well known fact that C^n is not a complete space if furnished with the norm (7) [2].

5. Possibility to Deliver from the Initial Conditions

In many cases one knows *a priori* what sort of disturbances acts on the system. It is then reasonable to choose the corresponding initial conditions for optimization. But often the perturbations are unknown and an optimization independent of the initial values is desired. We confine ourselves to the case when (4) is a linear system with constant coefficients. It can then be written as

$$\dot{x} = Ax, \tag{4a}$$

A being a real square $n \times n$ matrix with elements a_{ik}. The solution of this system with initial conditions (5) is [1]

$$x = e^{At}x_0, \tag{12}$$

x_0 is an element of the space R^n with norm $\|x_0\|^{R^n}$, x is from C^n with norm $\|x\|^{C^n}$. Depending of the norm we must eventually restrict x to a subspace of C^n to guarantee the existence of the norm. The expression (12) can be interpreted as a continuous linear transformation between the two spaces R^n and C^n, and e^{At} as a continuous linear operator. Such operators are themselves elements of a linear space, say $LC(R^n, C^n)$, where in the following way a norm is introduced [2]

$$\|e^{At}\|^{LC} = \sup_{x_0 \neq 0} \frac{\|e^{At}x_0\|^{C^n}}{\|x_0\|^{R^n}} = \sup_{\|x_0\|=1} \|e^{At}x_0\|^{C^n}. \tag{13}$$

Between the norms the inequality

$$\|x\|^{C^n} \leqq \|e^{At}\|^{LC} \|x_0\|^{R^n}$$

is valid.

For optimal design we minimize now the expression $\|e^{At}\|^{LC}$. From this follows an optimal set of parameters, the elements a_{ik} of A or functions of them, which depend of the norm in $LC(R^n, C^n)$, i.e. the norms in R^n and C^n, but no longer of the initial values x_0. Here the optimization can also be interpreted as an approximation of the 0-matrix in the space $LC(R^n, C^n)$. Since $\|e^{At}\|$ does not depend lineąrly of the a_{ik}, it is a nonlinear problem. Taking for x_0 and x the norms (2) with p=2 and (7) respectively, the norm in $LC(R^n, C^n)$ is

$$\|e^{At}\| = \left[\rho\left(\int_0^\infty e^{A^T t}e^{At}dt\right)\right]^{\frac{1}{2}}. \tag{14}$$

A^T is the transpose of A, $\rho(B) \equiv \max_i |\lambda_i|$ is the spectral radius of the matrix B whose eigenvalues are λ_i. Naturally we have to take for granted the existence of the integral in (14).

6. Conclusions

With the concept of vector space norms it is possible to give a systematic treatment of the optimal design problem of discrete systems. The choice of the adequate norm depends on the concrete problem and is a mainly technical question. Just the same must be said about the initial conditions to be chosen if one uses the method of section 4.

Finally we remark that the investigations of section 4 can be carried out in metric spaces, too. But since this does not lead to new aspects, we took the more familiar notion of normed linear spaces, also with regard to section 5.

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